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INFLUENCE OF HEAT TRANSFER ON MELTING AND SOLIDIFICATION IN FORCED FLOW

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Abstract—The solidification of fluids flowing along a plane wall or through a pipe is calculated with the assumption of a finite ambient heat transfer, and an imposed or known heat flux to the solid–liquid interface.

A very good approximation is obtained if in the solid phase a parabolic temperature distribution is assumed which satisfies all boundary conditions and agrees with the energy equation only at the solid–fluid interface. A comparison with measurements on an analog electrical model and with other numerical results shows that the error lies below 2·5 per cent.

In pipe flow the growth of the solid layer, for certain values of the ambient heat transfer and heat flux at the fluid–solid interface, may come to a stop at two critical points of which one designates stable conditions, the other an unstable state. It depends on the preceding history of the solid phase whether a stable or an unstable point is attained.

NOMENCLATURE

a,	thermal diffusivity;
c,	specific heat;
F_i,	inner surface area of pipe;
F_a,	outer surface area of pipe;
F_m,	logarithmic mean surface area of pipe;
f_0, f_1, f_2,	coefficient functions;
h,	latent heat of solidification;
k',	coefficient of heat transmission;
n,	parameter; $n = 0$ for plane wall problem; $n = 1$ for pipe flow;
$\dot{q}$,	heat flux density;
t,	time;
T_0,	initial Temperature;
T_s,	solidification temperature;
T_u,	ambient temperature, temperature of coolant
x,	coordinate normal to direction of flow;
x_0,	characteristic length, e.g. pipe radius;
X,	x/x_0, dimensionless coordinate normal to direction of flow;
Y,	$\int_1^X dX/X^n$, transformed coordinate;
z,	coordinate in direction of flow;
Z,	z/x_0, dimensionless coordinate in direction of flow.

Greek symbols

α,	coefficient of heat transfer;
δ_w,	wall thickness;
	$= \int_1^{\xi^*} dX/X^n$, transformed coordinate;
λ,	thermal conductivity;
ξ,	coordinate of solidified layer;
ξ^*,	$= \xi/x_0$, dimensionless coordinate of solid layer;
ρ,	density.

Subscripts

L,	in the flowing fluid (liquidus);
s,	in the solid (solidus);
w,	wall.

Dimensionless parameters

Bi,	$k' x_0/\lambda_s$, Biot number;
Ph,	$h/c_s(T_s - T_u)$, phase-conversion parameter;
q^*,	$\dot{q}x_0/(\rho_s h a_s)$, heat flux parameter;
τ,	$a_s t/x_0^2$, Fourier number;
ϑ,	$(T_2 - T)/(T_s - T_u)$, dimensionless temperature.

1. INTRODUCTION

IN NUMEROUS problems of conductive and of convective heat transfer there occur changes of phase accompanied by an absorption or a release of internal energy. Two of the oldest examples are the formation of ice and the solidification of lava streams. Technical examples are the sinking of mine pits by freezing-techniques, separation of mixtures by freezing out one or more components and solidification of liquid metals. Some of these problems have been treated under the following simplifying assumptions [1–6]. It was assumed that the liquid near the liquid–solid interface is at rest and at the solidification temperature. The majority of practical problems are not met by these assumptions. In many technical processes a liquid flows along a cooled surface and begins to solidify on this surface. Normally the average temperature of the liquid will be higher than solidification temperature, thus causing a flow of heat from the liquid to the solid phase and a decrease in the rate of solidification. If the heat flux is strong enough solidification will eventually cease or it may not even commence. Consider for instance the formation of ice on rivers and the fact that a river with rapid flow will take longer to freeze up than a slower one. The aim of the following analysis is to calculate speeds and duration of solidification for the described phenomena. The present investigation is limited to the flow along a plane wall and the flow in pipes which are technically of particular interest. As an essential boundary condition finite heat transfer between solidified layer and the surrounding shall be assumed. A boundary condition of constant wall temperature is satisfied by the limiting case of infinite heat transfer. Numerical calculations are based on a method of approximation originally due to Brovman and Surin [7] and extended by Megerlin [8] to problems of heat conduction with change of phase and which Megerlin showed to be superior in simplicity and accuracy to any approximations known as yet.

1.1. *Previous works*

The mathematical treatment of solidification processes by exact analytical methods proved to be fruitless due to high mathematical intricacy. The exact solutions due to Stefan [1] and Neumann [2] apply to more elementary problems, namely the solidification of a stationary liquid at the melting point temperature bounded by a plane wall at a constant temperature. Portnov [9] treated the same problem by expansion in series, permitting, however, arbitrary boundary conditions between the plane wall and its environment. The leading expressions in the series set forth by Portnov were recently evaluated by Westphal [10] in calculations concerning the thickness of arctic ice. The evaluation of only the first few expressions of the series is very cumbersome and can only be accomplished numerically.

Numerical methods [11–15] become very complex due to the nonlinear boundary condition for heat flux at the phase interface. The usual procedure is to estimate the thickness of the solid layer, then to determine the temperature distribution, which in turn provides a new value for the thickness of the layer. This is repeated until the estimated and the calculated value are in close enough agreement. As with most numerical methods one does not obtain results which explicitly interpret the influence of various parameters on solidification.

Among the analytical approximations the integral methods by Goodman [16–18] and the variation method developed by Biot [19, 20] in dealing with the temperature development in flight structures have proved successful in problems of heat conduction without changes of state. Goodman's integral methods is based on the method by von Kármán and Pohlhausen, familiar in boundary-layer theory. The energy equation is satisfied only on the average and the temperature distribution is represented by a second degree polynomial. The evaluation of the integral presents considerable difficulties, however, when dealing with problems of solidification with finite heat transfer at the wall. This difficulty also arises when the variational method is employed. According to investigations by Megerlin [8] the Goodman method does not yield very accurate results in problems of melting and solidification.

Solidification in flowing liquids has been investigated as yet only for specific assumptions. Libby and Chen [21] applied Goodman's integral method to investigate the sedimentation of precipitants from a gas stream. Lapadula and Müller [22] treated the same problem by Biot's variation method, and Beaubouef and Chapman [15] by a method of numerical integration. Basic assumptions in these investigations were: solidification on a plane wall of uniform constant temperature and constant heat flux from the flowing liquid to the solid phase. The results of these calculations appear as a special case of the solutions for the plane wall presented in the following.

2. MATHEMATICAL FORMULATION

Consider the flow of a liquid through a duct bounded by solid walls or along a plane wall as shown in Fig. 1. The thickness of the wall is δ_w, its heat conductivity be λ_w. The outside of this wall is cooled by some liquid. Let the temperature T_u of this coolant be lower than solidification temperature T_s of the fluid flowing within the duct or along the plane wall. This latter temperature is assumed to be initially at $T_0 > T_s$. The heat-transfer coefficient between coolant and the wall is α. As shown in Fig. 1, the liquid will solidify and the thickness $x_0 - \xi$ of the solidified layer is generally a function of time and position along the wall. The rate of solidification is essentially governed by the flux of heat into the solid layer and the heat transfer to the coolant.

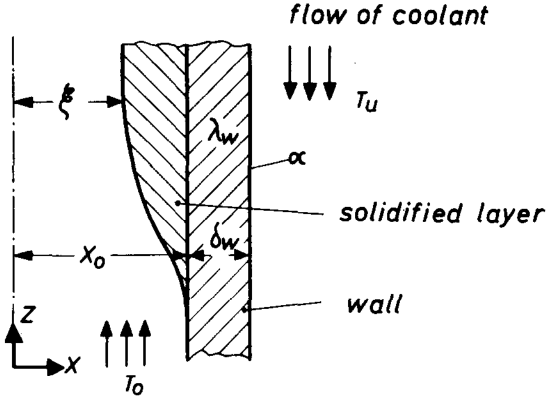


FIG. 1. Solidification of liquid on a wall.

The calculation of the rate of solidification will be made with the following assumption: the solid layer is thin compared with its extension in the direction of flow so that the flow of heat in this direction is negligibly small compared to that normal to the flow. There exists a well defined fluid–solid boundary, and the physical properties of the solid and the fluid are different but not a function of temperature.† The energy equation for the solid phase then is:

$$\frac{\partial T}{\partial t} = a_s[\partial^2 T/\partial x^2 + (n/x)\,\partial T/\partial x] \tag{1}$$

with $n = 0$ for the (one-dimensional) plane wall problem, $n = 1$ for the cylindrical case and $n = 2$ for the spherical case. At the solid–liquid interface the energy balance is

$$\rho_s h(\partial\xi/\partial t) = \lambda_s(\partial T/\partial x)_\xi + \dot{q}. \tag{2}$$

The symbols are: t for the time, ρ_s for the density of the solid phase, h for its enthalpy of fusion, $\dot{q}$ for the flux density of heat transferred from the fluid stream to the solid phase, λ_s for the coefficient of heat conductivity and a_s for the thermal diffusivity of the solid phase. The coordinate ξ of the solid phase is a function of time t and the coordinate z.

† Principally the present method is applicable also to cases of temperature-dependent properties [8].

Equations (1) and (2) must be solved with the following boundary conditions:

$$T(x = \xi) = T_s \tag{3}$$

$$-\lambda_s(\partial T/\partial x)_{x_0} = k' \, . \, [T(x = x_0) - T_u]. \tag{4}$$

The coefficient of heat transmission k' between solid phase and coolant is given by

$$k' = (\delta_w F_i)/(\lambda_w F_m) + F_i/(\alpha F_a). \tag{5}$$

F_i is the area of the wall bounding the solid phase and F_a that bounding the coolant. The mean wall area F_m of a pipe is

$$F_m = (F_a - F_i)/\ln (F_a/F_i), \tag{6}$$

that of a plane duct is

$$F_m = F_i = F_a. \tag{7}$$

For the mathematical treatment dimensionless parameters are introduced which are defined as follows:

the dimensionless temperature	$\vartheta = (T_s - T)/(T_s - T_u)$,
the dimensionless time	$\tau = a_s t/x_0^2$,
the dimensionless coordinates	$X = x/x_0 ; \xi^* = \xi/x_0$,
the Biot number	$Bi = k'x_0/\lambda_s$
the phase-conversion parameter	$Ph = h/[c_s(T_s - T_u)]$
and the heat-flux parameter	$q^* = \dot{q}x_0/(\rho_s \, . \, h \, . \, a_s)$.

Finally, by introducing the coordinate transformations

$$Y = \int_1^X \frac{\mathrm{d}X}{X^n} \quad \text{and} \quad \eta = \int_1^{\xi^*} \frac{\mathrm{d}X}{X^n}, \tag{8}$$

the energy equations (1) and (2) and the boundary conditions (3) and (4) become:

$$\partial\vartheta/\partial\tau = (1/X^{2n}) \, . \, \partial^2\vartheta/\partial Y^2 \tag{9}$$

$$\partial\xi^*/\partial\tau = -(1/Ph) \, . \, (\partial\vartheta/\partial X)_{\xi^*} + q^* \tag{10}$$

$$\vartheta(\xi^* ; \tau) = 0 \tag{11}$$

$$\partial\vartheta(X = 1 ; \tau)/\partial X = Bi[1 - \vartheta(X = 1 ; \tau)]. \tag{12}$$

This last boundary condition becomes $\vartheta(X = 1 ; \tau) = 1$ if the wall temperature $T(x = x_0 ; t)$ is specified to be constant. This is the case when the Biot number is very large, $Bi \to \infty$. The flux of heat $\dot{q}$ and accordingly the heat-flux parameter q^* are functions of the coordinates ξ and z and of ξ^* and $z = z/x_0$ respectively. The determination of such a function is not the goal of the following analysis. Since the thickness of the solid phase varies with time, the flowing liquid will normally be accelerated as the layer grows. Therefore, in order to determine $q(\xi, z)$, in addition to the above equations the Navier–Stokes equation, the continuity equation and the energy equation for the flowing liquid (i.e. together a system of at least five coupled partial differential equations, some of them nonlinear), would have to be solved. In the following discussion the function $q(\xi, z)$ may have any arbitrary form, provided it is specified.

3. SOLUTION

3.1. *Stationary solution*

If the solidification enthalpy of the fluid is large and the energy stored in the solid layer is comparatively small the change of temperature with time may be considered negligibly slow. This condition is met if the phase-conversion parameter Ph is sufficiently large. The energy equation (9) simplifies to an equation for stationary heat conduction:

$$\partial^2\vartheta/\partial Y^2 = 0. \tag{13}$$

Its general solution

$$\vartheta = f_0 + f_1 Y \tag{14}$$

will be called the stationary solution in the following. Due to boundary condition (11) at the phase interface, this solution becomes:

$$\vartheta = f_1(Y - \eta). \tag{15}$$

The function f_1, determined by condition (12) at the cooling surface is

$$f_1 = Bi/(1 - Bi\eta). \tag{16}$$

Introducing this and the fact that

$$\frac{\partial\vartheta(\xi^*;\tau)}{\partial X} = \left(\frac{\partial\vartheta}{\partial Y}\right)_\eta \left(\frac{\partial Y}{\partial X}\right)_{\xi^*} = \left(\frac{\partial\vartheta}{\partial Y}\right)_\eta \frac{1}{\xi^{*n}} = f_1\frac{1}{\xi^{*n}} \tag{17}$$

into the energy equation (10) renders an ordinary differential equation for the rate of solidification:

$$-\xi^{*n}\left(\frac{\partial\xi^*}{\partial\tau}\right) = Bi/[Ph(1 - Bi\eta)] - q^*\xi^{*n}. \tag{18}$$

Integration renders the time or the extent of solidification. For the plane wall problem (one-dimensional, $n = 0$) this is

$$\tau = \int_{\xi^*}^{1} \frac{1 + Bi(1 - \xi^*)}{Bi/Ph - q^* - q^*Bi(1 - \xi^*)}\,d\xi^*. \tag{19}$$

Integration is only possible if the function $q^*(\xi^*, z)$ is known.

Supposing the latter is independent of ξ^*, the result is

$$\tau = -\frac{1 - \xi^*}{q^*} - \left(\frac{1}{q^*Bi} + \frac{(1/Ph) - (q^*/Bi)}{q^{*2}}\right) \times \ln\left[\frac{(1/Ph) - (q^*/Bi) - q^*(1 - \xi^*)}{(1/Ph) - (q^*/Bi)}\right]. \tag{20}$$

This relation can be simplified by suitably disposing of the characteristic length x_0. Since in the case of solidification on a plane wall there exists no characteristic length of any preference one can randomly put $Bi = 1$, thus obtaining a characteristic length $x_0 = \lambda_s/k'$. Equation (20) then simplifies to

$$\frac{\tau}{Ph} = \frac{1 - \xi^*}{q^*Ph} - \frac{1}{(q^*Ph)^2}\ln\frac{1 - q^*Ph(2 - \xi^*)}{1 - q^*Ph} \tag{21}$$

where now $x_0 = \lambda_s/k'$ and q^* may only vary with z and not with ξ^*.

Equation (20) furthermore renders a simple expression for the case of constant wall temperature. One must deal with the limiting case of $Bi \to \infty$. If the characteristic length is chosen so that $q^* = 1$, one finds

$$\frac{\tau}{Ph} = -\frac{1 - \xi^*}{Ph} - \frac{1}{Ph^2}\ln[1 - Ph(1 - \xi^*)], \tag{22}$$

where now $x_0 = (\rho_s h a_s)/q$

For the case of pipe flow ($n = 1$) integration of equation (18) renders

$$\frac{\tau}{Ph} = \int_1^{\xi^*} \frac{\xi^*(1 - Bi\ln\xi^*)}{Phq^*\xi^*(1 - Bi\ln\xi^*) - Bi}\,d\xi^*. \tag{23}$$

Here the pipe radius is the characteristic length: $x_0 = R$.

In case of constant wall temperature ($Bi \to \infty$) this becomes

$$\frac{\tau}{Ph} = \int_1^{\xi^*} \frac{\xi^*\ln\xi^*}{Phq^*\xi^*\ln\xi^* + 1}\,d\xi^*. \tag{24}$$

By numerical integration one obtains the time of solidification.

3.2. *Transient solution*

The phase-conversion parameter Ph is often small. In such cases the energy stored in the solid layer may no longer be neglected with respect to the melting enthalpy and hence, in the energy equation (9) the change of temperature with time may not be considered neglegibly slow. An exact analytical treatment of the problem is no longer possible. One must either employ numerical procedures or search for analytical approximations. The latter have the advantage that they show, better than numerical methods, the influence of individual parameters. Therefore, such an analytical approach shall again be persued.

A parabolic expression in $(Y - \eta)$ is chosen to represent the temperature distribution.†

$$\vartheta = f_1(\tau).(Y - \eta) + f_2(\tau).(Y - \eta)^2. \tag{25}$$

† The expression that Megerlin [8] denoted as the second approximation is included herein. It is obtained if $Y - \eta$ is calculated for the one-dimensional problem, i.e. by putting $Y - \eta = X - \xi^*$.

It directly satisfies the boundary condition $\vartheta(\xi^*;\tau) = \vartheta(\eta;\tau) = 0$. The chosen expression contains the three unknown functions $f_1(\tau)$, $f_2(\tau)$ and $\eta(\tau)$. To calculate these, equations (9, 10, 12) are available. Since the above expression does not represent a general solution of the energy equation (9), the latter can only be satisfied at specific points $(Y - \eta)$ if equations (10) and (12) are to be satisfied simultanuously. It is suggested that the energy equation be satisfied at the phase interface $Y = \eta$, because the resulting temperature distribution then is particularly accurate in this vicinity. This should be quite relevant, particularly for the derivation of the rate of solidification, which strongly depends on the internal energy per unit volume stored near the interface and not so much on the energy stored in more distant volume elements. As a matter of fact, the Goodman method, where the heat-conduction equation is only averagely satisfied, produces less accurate results than if the latter is satisfied by the above expression merely in the vicinity of the phase interface. Megerlin showed by comparison with the Neumann example that application of the Goodman method causes a maximum error of 7·5 per cent, whereas application of the above parabolic expression brings a maximum of 2·5 per cent.

If expression (25) is substituted into energy equation (9) the latter is satisfied at the phase interface if the functions defining the coefficients are correlated by

$$f_2 - -\tfrac{1}{2}f_1\xi^{*n}(\partial\xi^*/\partial\tau). \qquad (26)$$

With condition (12) at the cooling surface and correlation (26) one onbtains

$$f_1 = Bi/[1 - Bi\eta + \tfrac{1}{2}\xi^{*n}(\partial\xi^*/\partial\tau)\eta(2 - Bi\eta)]. \qquad (27)$$

To finally describe the growth $\xi^*(\tau)$ or $\eta(\tau)$ of the solid phase there remains the energy balance (10) at the solid–fluid interface. By introducing the coefficient functions f_1 and f_2, which are now available, into equation (10) one finds for the growth of the solid layer the ordinary nonlinear differential equation

$$\frac{\partial\xi^*}{\partial\tau}(\xi^*)^n = \frac{-Bi/Ph}{1 - Bi\eta + \tfrac{1}{2}\xi^{*n}(\partial\xi^*/\partial\tau)\eta(2 - Bi\eta)} + q^*\xi^{*n}. \qquad (28)$$

Thus, for the plane wall problem ($n = 0$):

$$\frac{\partial\xi^*}{\partial\tau} = \frac{-Bi/Ph}{1 - Bi\eta + \tfrac{1}{2}(\partial\xi^*/\partial\tau)\eta(2 - Bi\eta)} + q^*. \qquad (29)$$

For pipe flow ($n = 1$):

$$\frac{\partial\xi^*}{\partial\tau}.\xi^* = \frac{-Bi/Ph}{1 - Bi\eta + \tfrac{1}{2}\xi^*(\partial\xi^*/\partial\tau)\eta(2 - Bi\eta)} + q^*\xi^*. \qquad (30)$$

A similar expression is obtained for the sphere if $n = 2$ is substituted. The rates of solidification in the plane wall problem, in the cylindrical and in the spherical problem thus appear closely related. This correlation is defined by the following rule:

To calculate, from the rate of solidification $(\partial\xi^*/\partial\tau)$ in the one-dimensional problem, the corresponding rates in the cylinder or sphere problem one must replace $(\partial\xi^*/\partial\tau)$ by $(\partial\xi^*/\partial\tau)\xi^{*n}$ and the heat flux parameter q^* by $q^*\xi^{*n}$ with $n = 1$ for the cylinder and $n = 2$ for the sphere. The parameter η, according to its definition, equation (8), is formed differently for the plane wall problem, the cylinder and the sphere.

This rule is also true if a linear function in $Y - \eta$ is assumed for the temperature distribution. It does not hold, however, if an expression of third or higher degree is taken. For it follows from equations (27) and (28) that in cases of a parabolic distribution, the coefficient function f_1 depends only on the parameter η, not however, on n. Only if $f_1 = (\partial\vartheta/\partial Y)_\eta$ is solely a function of η does the above rule follow from the energy balance equation (10) at the phase interface. Then

$$(\partial\vartheta/\partial x)_{\xi^*} = (\partial\vartheta/\partial Y)_\eta/\xi^{*n} = f_1(\eta)\xi^{*n}$$

and the energy balance becomes

$$(\partial\xi^*/\partial\tau)\xi^{*n} = -\frac{1}{Ph}\cdot f_1(\eta) + q^*.\xi^{*n} \qquad (31)$$

from which the above rule becomes directly apparent. If a polynomial of higher than second degree is used for the temperature distribution and is substituted in the energy equation (9), a comparison of the coefficients of equal powers of $(Y - \eta)$ shows that f_1 no longer is a function of η alone, but also of n. The simple rule above, concerning the relation between the rate of solidification in the one-dimensional problem and that in the cylindrical and spherical problem are thus only approximately rather then rigorously true.†

3.2.1. *The rate of solidification and its limiting values*. The rate of solidification when solidification commences ($\xi^* = 1$), calculated by equations (29) and (30) for the one-dimensional problem and for pipe flow, is

$$\partial\xi^*/\partial\tau = -Bi/Ph + q^* \tag{32}$$

or, introducing parameters with the original dimensions:

$$\rho_s h\, \partial\xi/\partial t = -k'(T_s - T_u) + \dot{q}. \tag{32a}$$

Towards the end of solidification ($\xi^* \to 0$) in pipe flow one has $\partial\xi^*/\partial\tau \to -\infty$. The rate of solidification in a pipe increases rapidly at the end of the solidification period because the volume of the cylindrical layers becomes increasingly smaller.

Solidification stops when $\partial\xi^*/\partial\tau = 0$. It follows from equation (29) for the plane wall problem that then

$$q^* = Bi/[Ph(1 - Bi\eta)] \tag{33}$$

from which the thickness $\delta = x_0 - \xi$ of the solid layer is found to be

$$\delta = \frac{\lambda_s(T_s - T_u)}{\dot{q}} - \frac{\lambda}{k'} \tag{33a}$$

and the heat flux

$$\dot{q} = (T_s - T_u)\left(\frac{\delta}{\lambda_s} + \frac{1}{k'}\right)^{-1}. \tag{33b}$$

This result can be plausibly explained. It represents the fact that solidification comes to a stop when the flux of heat transferred by convection at the phase interface exactly equals that transferred to the coolant at the wall, and thus no enthalpy of solidification is released.

If the rate of solidification for the plane wall problem, according to equation (29), is plotted over the thickness of the solid layer, the result is a set of curves, as shown in Figs. 2 and 3,

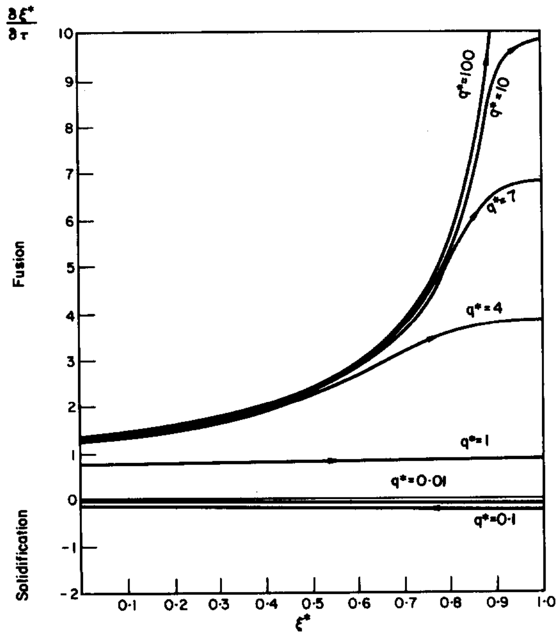


FIG. 2. Rate of solidification of water with $Ph = 5{\cdot}401$ in the plane wall problem.

with the parameter q^* which is assumed constant in both diagrams. Figure 2 was plotted for water with $Ph = 5{\cdot}401$, whereas Fig. 3 shows solidification rates for steel with $Ph = 0{\cdot}271$. Positive values of $\partial\xi^*/\partial\tau$ designate melting of solid, negative values designate solidification of fluid. Both diagrams show that solidification comes to a stop as the solid layer grows thicker.

† This is also true for the theorem derived by Lin [23] which follows as a special case from the rule above when $q^* = 0$. Lin's proof is based on the assumption that f_1 (C_1 in Lin's work) always is merely a function of the parameter η. Lin comes to this supposition because, in comparing the coefficients (at the top of p. 10 in his work [23]) he apparently overlooked the fact that the factor $(X/\xi)^{2n}$ depends on $(Y - \eta)$ and on powers of this parameter.

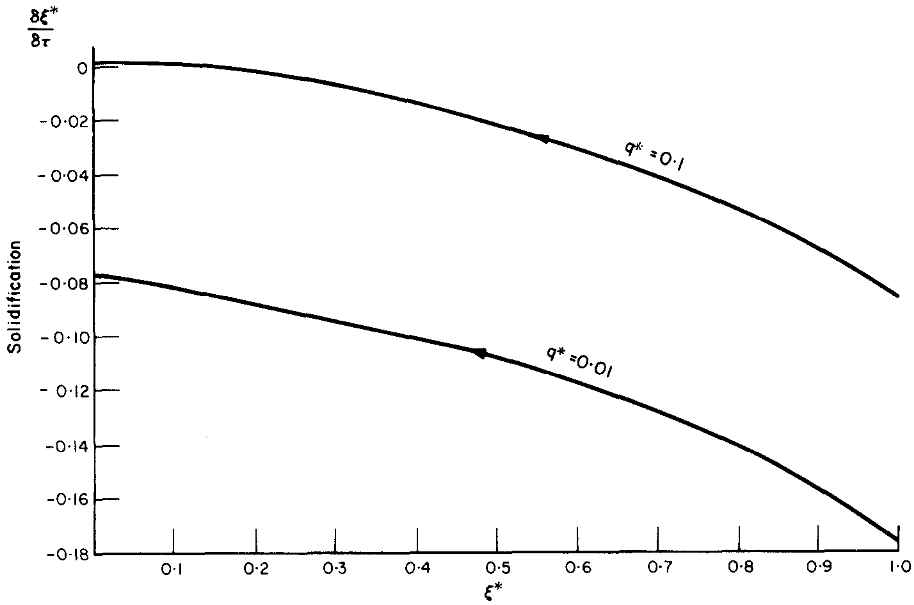


FIG. 3. Rate of solidification of water with $Ph = 5{\cdot}401$ in the plane wall problem.

In pipe flow solidification ceases, as equation (30) shows, when

$$\xi^*(1 - Bi \ln \xi^*) = Bi/(Phq^*). \tag{34}$$

In order to discuss the roots of this expression consider the curve $\bar{Y}(\xi^*) = \xi^*(1 - Bi \ln \xi^*)$. The latter has a relative maximum at $\xi^* = \exp[(1/Bi) - 1]$ and $\bar{Y} = Bi \exp[(1/Bi) - 1]$. The roots are the points of intersection of the curves $\bar{Y}(\xi^*)$ and the straight line $\bar{Y} = Bi/Phq^*$. As Fig. 4 suggests, there exist, depending on the magnitude of the Biot number and the parameter $Bi/Phq^* = k'(T_s - T_u)/\dot{q}$, either none, one or two roots. There is one root if $Bi/Phq^* \leqslant 1$, there are two roots if $Bi > 1$ and $1 \leqslant Bi/Phq^* \leqslant Bi \exp[(1/Bi) - 1]$ and there exists no root if $Bi/Phq^* > Bi \exp[(1/Bi) - 1]$.

It is surprising that apparently under certain conditions solidification may come to a stop at two different positions. Figs. 5–7 will serve to explain this remarkable behaviour.

In these the rate of solidification $\partial\xi^*/\partial\tau$ is plotted over the coordinate ξ^*. Positive values of $\partial\xi^*/\partial\tau$ represent melting of the solid, negative values represent solidification of the fluid

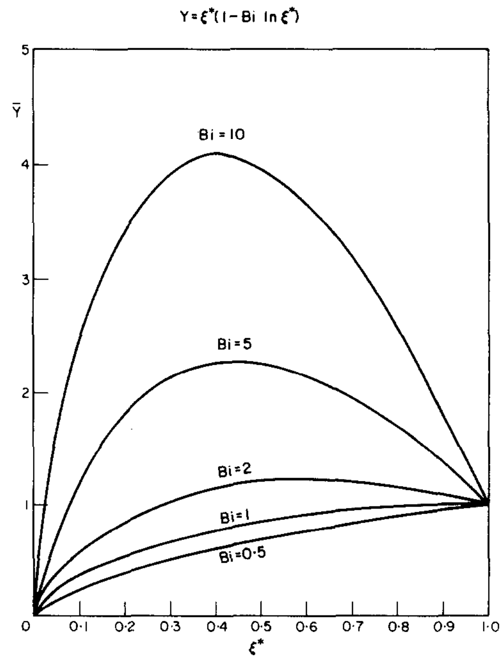


FIG. 4. Points where solidification ceases in pipe flow.

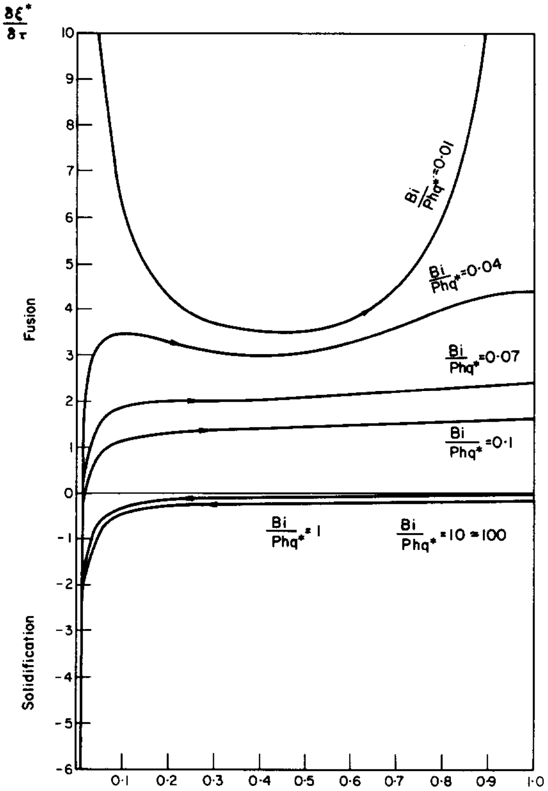


FIG. 5. Rate of solidification of water with $Ph = 5{\cdot}401$ in pipe flow. $Bi = 1$.

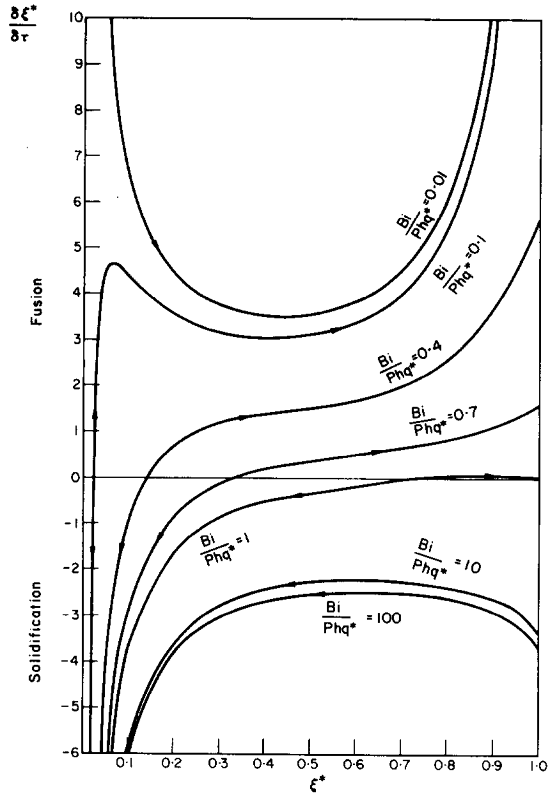


FIG. 6. Rate of solidification of steel with $Ph = 0{\cdot}271$ in pipe flow. $Bi = 1$.

phase, as indicated by arrows along the curves. Figure 5 was plotted for water with $Ph = 5{\cdot}401$, Figs. 6 and 7 for steel with $Ph = 0{\cdot}271$. Those curves for which $Bi/Phq^* \leqslant 1$, have one root which evidently signifies some unstable state. On the left of the ξ^*-axis intersection the arrows along the curve indicate that a solid layer, once it exists, will continue to grow. On the right of such an intersection the solid layer will melt away. If the pipe was initially free of solid phase no liquid will solidify because, as $k'(T_s - T_u)/\dot{q} \leqslant 1$, the heat flux coming from the core of the flow is greater than or at least equal to that removed through the wall of the pipe.

If $Bi \leqslant 1$ whilst also $Bi/Phq^* \geqslant 1$ then, as Figs. 5 and 6 show, all liquid will solidify. This is the case also if $Bi > 1$ and $Bi/Phq^* \geqslant Bi \exp[(1/Bi) - 1]$. The rate of solidification passes through a stage of minimum speed.

If $Bi > 1$ whilst also

$$1 < Bi/Phq^* < Bi \exp[(1/Bi) - 1]$$

the liquid in a pipe will begin to solidify. As Fig. 7 suggests, the solid layer will stop growing at such a thickness which corresponds to the first intersection of the ξ^*-axis closest to $\xi^* = 1$. This position is stable (i.e. if the layer should slightly grow or melt due to some disturbance in heat flux, it will return to its original thickness when the disturbance is gone). Should, however, the solid layer have grown, for instance by intense initial cooling, to the intersection point on the far left of Fig. 7, then

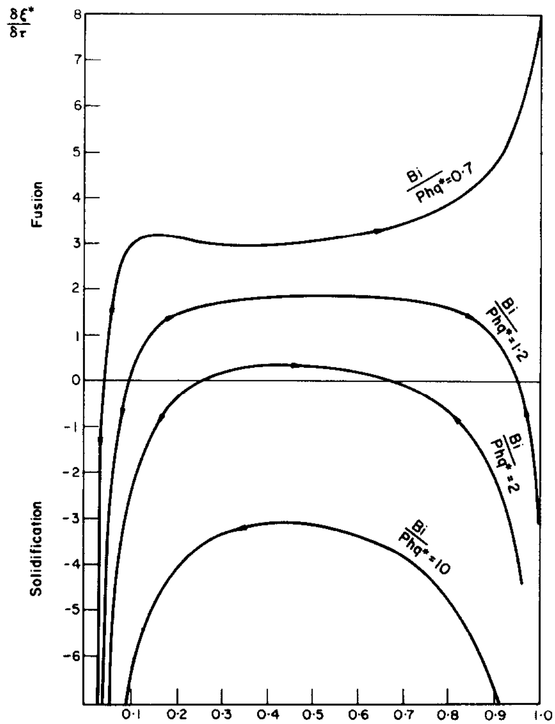


FIG. 7. Rate of solidification of steel with $Ph = 0{\cdot}271$ in pipe flow. $Bi = 5$.

either further solidification or melting can occur, depending on whether the interface layer is caused to grow or to melt by some disturbance. Once melting begins it will not stop until the stable intersection position on the right is reached.

These considerations show that the behaviour of the solid layer depends on its preceding history. Thus, under equal conditions but with different preceding history, either solidification or melting can occur. Table 1 is a summary of these varied behaviours solidification in pipes can have.

3.2.2. *Solidification time*

The time for solidification is obtained by integrating the equations for the rate of solidification (29) and (30).

For the one-dimensional problem ($n = 0$) and constant wall temperature ($Bi \to \infty$) it follows from equation (29), if the characteristic length x_0 is chosen so that $q^* = 1$, that

$$\tau = \int_1^{\xi^*} \frac{(1 - \xi^*)\,.\,d\xi^*}{X_0 - \sqrt{\{X_0^2 + 2[1/Ph - (1 - \xi^*)]\}}} \tag{35}$$

Table 1. Solidification of flowing liquids in pipes

Characteristic parameters		Behaviour of the liquid
Bi	$Bi/Phq^* = k'(T_s - T_w)/\dot{q}$	
$\gtrless 1$	<1	(a) No solidification if initially there existed no solid layer (b) Fusion if an initially existing layer was $\xi^* > \xi_1^*$[ξ_1^* from equation (34)]. (c) Solidification if an initially existing layer was $\xi^* < \xi_1^*$ [ξ_1^* from equation (34)].
$\leqslant 1$ $\geqslant 1$	$\geqslant 1$ $\geqslant Bi \exp[(1/Bi) - 1)]$	The liquid solidifies completely
$\geqslant 1$	$1 \leqslant Bi/Phq^* \leqslant Bi \exp[(1/Bi) - 1)]$	(a) Solidification to a thickness ξ_1^* if initially there existed no solid layer. [ξ_1^* is the root closest to $\xi^* = 1$ in equation (34).] (b) All liquid solidifies if an initially existing solid layer was $\xi^* < \xi_0^*$. [ξ_0^* is the root closest to $\xi^* = 0$ in equation (34).] (c) Fusion to a point ξ_1^* if an initially existing solid layer was $\xi^* > \xi_0^*$. [ξ_1^{*1} is the root closest to $\xi^* = 1$, ξ_0^* is the root closest to $\xi^* = 0$ in equation (34).]

with

$$X_0 = 1 + \tfrac{1}{2}(1 - \xi^*) \quad \text{and} \quad x_0 = \rho_s h a_s / \dot{q}.$$

For the one-dimensional problem ($n = 0$) and finite heat transfer at the wall the characteristic length is chosen to make $Bi = 1$. Integration of equation (29) yields

$$\tau = \int_1^{\xi^*} \frac{(1 - \xi^*)(3 - \xi^*) \, . \, d\xi^*}{X_1 - \sqrt{\{X_1^2 + 2[1/Ph - q^*(2 - \xi^*)](1 - \xi^*)(3 - \xi^*)\}}} \tag{36}$$

with $X_1 = 2 - \xi^* + \frac{1}{2}q^*(1 - \xi^*)(3 - \xi^*)$ and $x_0 = \lambda/k'$.

For pipe flow ($n = 1$) and constant wall temperature ($Bi \to \infty$) one obtains from equation (30)

$$\tau = \int_1^{\xi^*} \frac{\xi^*(\ln \xi^*)^2 \, . \, d\xi^*}{X_2 - \sqrt{\{X_2^2 + 2[1/Ph + q^*\xi^* \ln \xi^*](\ln \xi^*)^2\}}} \tag{37}$$

with $X_2 = -\ln \xi^* + (\tfrac{1}{2})q^*\xi^*(\ln \xi^*)^2$ and $x_0 = R$.

For pipe flow ($n = 1$) with finite heat transfer at the wall integration of equation (30) yields

$$\tau = -\int_1^{\xi^*} \frac{\xi^* \ln \xi^*(2 - Bi \ln \xi^*) \, . \, d\xi^*}{X_3 - \sqrt{\{X_3^2 + 2[Bi/Ph - q^*\xi^*(1 - Bi \ln \xi^*)](-\ln \xi^*)(2 - Bi \ln \xi^*)\}}} \tag{38}$$

with $X_3 = 1 - Bi \ln \xi^* - \frac{1}{2}q^*\xi^* \ln \xi^*(2 - Bi \ln \xi^*)$ and $x_0 = R$.

With the assumption of constant values of q^* and numerical integration by the Simpson rule, the values listed in Table 2 were obtained for two numerical examples. For comparison the results obtained by the stationary solution are also listed. When the phase-conversion parameter Ph is high, stationary and instationary solution differ only slightly, as shown by the example with $Ph = 5{\cdot}401$ for the freezing of water. They differ considerably, however, if the phase-conversion parameter is small, as one can see in the example with $Ph = 0{\cdot}271$ for the solidification of steel. Thus, for small values of the phase-conversion parameter the stationary solution is rather unsatisfactory.

Figures 8 and 9 show various solidification curves. They show that if solidification comes to a stop the solid–liquid interface approaches this final position asymptotically.

4. ATTAINED ACCURACY AND COMPARISON WITH RESULTS BY OTHER AUTHORS

A correct estimation of errors due to the instationary solution with parabolic temperature distribution in the solid phase, which was mainly applied here, has not succeeded as yet. As Megerlin [8] found by comparison with the

Table 2. Solidification times, numerical results for the one-dimensional problem ($n = 0$); $q^* = 0{\cdot}1$

$1 - \xi^*$	$Ph = 5{\cdot}401$		$Ph = 0{\cdot}271$	
	τ	$\tau_{\text{stationary}}$	τ	$\tau_{\text{stationary}}$
0·01	0·1184	0·1187	0·002850	0·002799
0·02	0·2396	0·2401	0·005824	0·005627
0·03	0·3637	0·3641	0·008916	0·008484
0·04	0·4908	0·4908	0·01212	0·011369
0·05	0·6209	0·6204	0·01544	0·01428
0·06	0·7542	0·7529	0·01886	0·01723
0·07	0·8906	0·8883	0·02238	0·02020
0·08	1·0303	1·0267	0·02600	0·02320
0·09	1·1734	1·1683	0·02972	0·02623
0·1	1·3199	1·3130	0·03353	0·02929
0·2	2·9962	2·9568	0·07657	0·06145
0·3	5·1459	5·0421	0·12779	0·09651
0·4	7·9576	7·7462	0·18658	0·1345
0·5	11·7637	11·3817	0·25258	0·1754
0·6	17·2365	16·5797	0·32558	0·2192
0·7	26·1057	24·9638	0·40543	0·2660
0·8	46·2672	43·9395	0·49202	0·3158

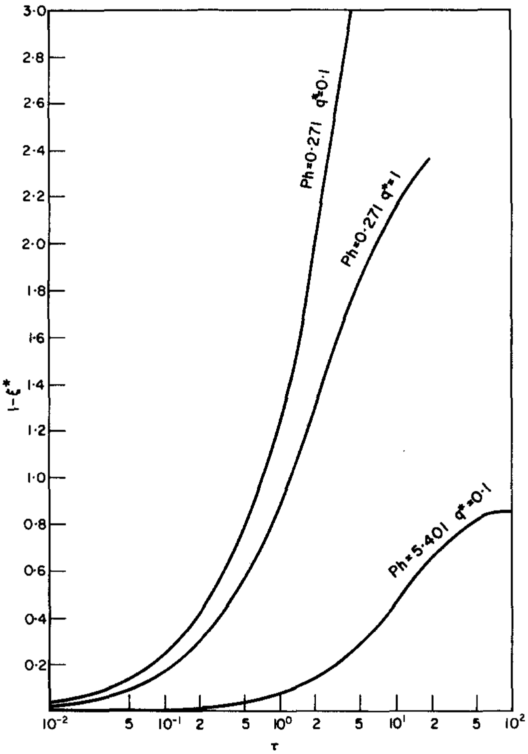


FIG. 8. Time for solidification in the plane wall problem. Transient solution.

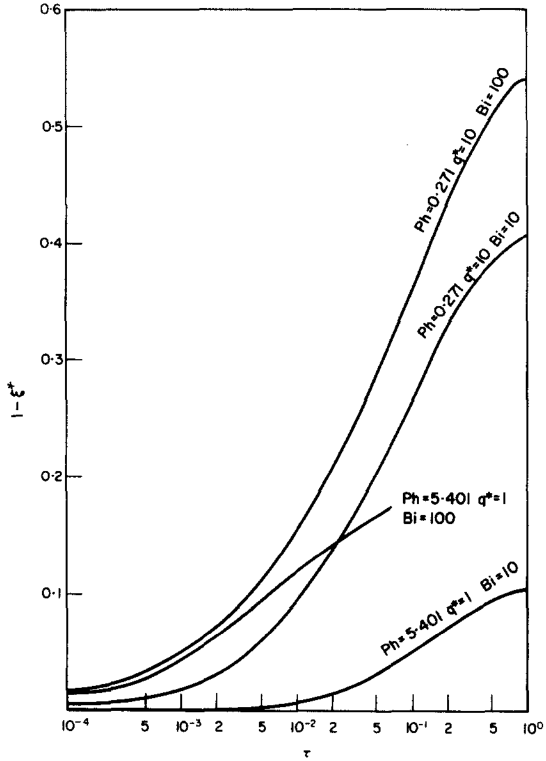
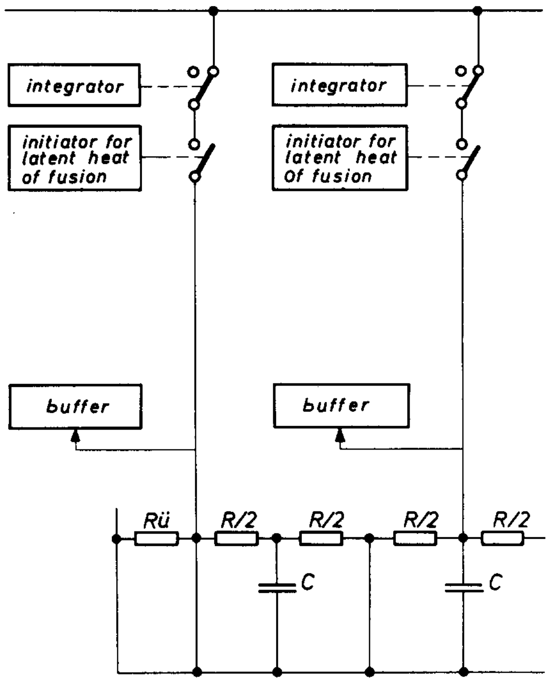


FIG. 9. Time for solidification in pipe flow. Transient.

Neumann problem, the error, depending on the magnitude of the phase-conversion parameter *Ph*, will be less than 2·5 per cent. The one-dimensional problem investigated in that work included the Neumann problem, $Bi \to \infty$, as a special case. This case has the strongest curvature of the temperature profile. In the other limiting case, $Bi \to 0$, the solidification temperature exists in the entire solid phase, i.e. the temperature profile is no longer curved. One may assume, therefore, that with finite values of the Biot number the parabolic approximation will cause errors lower than 2·5 per cent. To verify this the process of solidification was simulated by an analog electrical model. The solid phase is subdivided into a finite number of volume elements, each of which in the model is represented by an electrical resistor and a capacitor corresponding to its thermal resistance and its heat capacity. At the solid–liquid interface, analogous to the enthalpy of solidification, a current is introduced at constant voltage. Solidification of one volume element is completed when the product of current and time has reached a certain value which is proportional to the melting enthalpy of that volume element. A circuit diagram for such an electrical model is shown in Fig. 10. To simplify the problem the heat flux at the phase interface was put $q = 0$. By making the subdivisions sufficiently small it is quite possible, as previous investigations showed [24] and also was verified by comparison with the Neumann problem, to reduce the error in the thickness of the solid layer to below 1 per cent. According to Fig. 11 the differences between measurements on the



Rü = transfer resistance, R = resistance of one volume element, C = capacity of one volume element.

FIG. 10. Electrical simulation of solidification processes.

model and the parabolic approximation lie within the accuracy of the diagram. The error in the thickness of the solid layer is therefore certainly lower than 2·5 per cent. The one-dimensional problem was recently investigated by Beaubouef and Chapman [15] for the assumption of constant wall temperature ($Bi \to \infty$) and time-independent heat flux at the liquid–solid interface. With the transformation $\xi = Y/(x_0 - \xi)$ Beaubouef and Chapman rearranged the coupled differential equations (1) and (2) and solved them numerically for certain values of the phase-conversion parameter by iteration with a modified fourth-order Runge–Kutta procedure. Comparison of the results by Beaubouef and Chapman and those by equations (22) and (35) renders excellent agreement, as Fig. 12 shows. The results of the stationary solution by equation (22) are slightly displaced towards shorter times since the assumption of a stationary temperature distribution is only justified if the energy stored in the solid layer is negligibly small. In reality there is always some energy stored in the solid layer. The

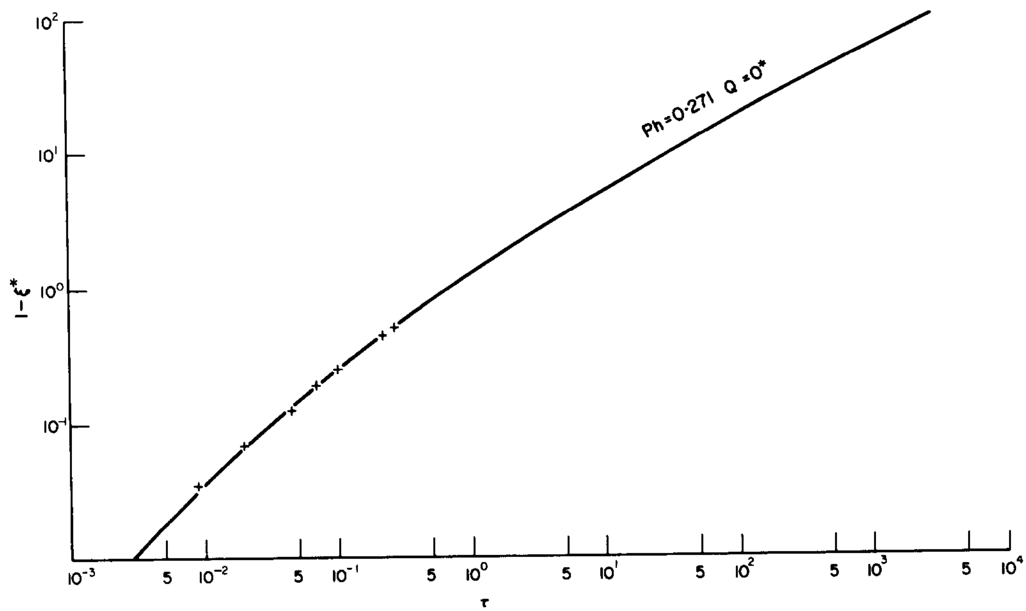


FIG. 11. Comparison of analytical results with measurements on the electrical model.

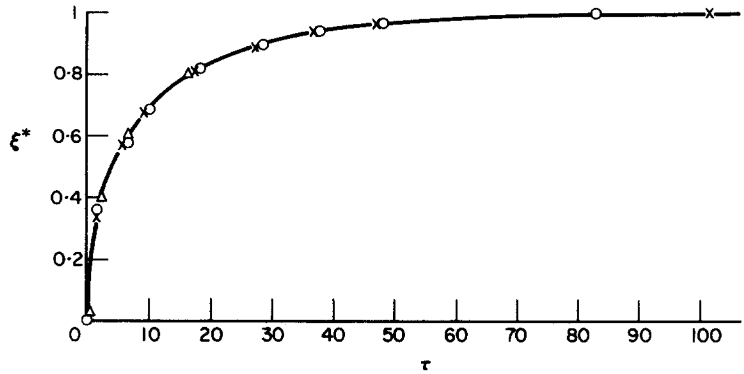


FIG. 12. Comparison with results by Beaubouef and Chapman [15],

solid layer, therefore, grows slower than the stationary solution is apt to suggest.

5. SUMMARY

Solidification of fluids flowing past a cooled, plane wall or through a pipe was analytically investigated. The mean temperature of the fluid was assumed higher than its melting temperature, thus causing a flux of heat into the solid phase. This heat flux will generally vary with time and location. The thermal resistance between the solid phase and the surrounding coolant was permitted to have any finite value, and consists of the thermal resistance of the wall and the thermal resistance between the wall and coolant. The problem leads to nonlinear, coupled, partial differential equations for which there is no exact solution. A first approximation was obtained by assuming a stationary temperature distribution in the solid phase, a better approximation was obtained with the assumption of a parabolic, transient temperature distribution. The latter satisfies all boundary conditions, but satisfies the heat-conduction equation only at the solid–liquid interface.

A comparison with measurements obtained by electrical simulation and with numerical results by other authors showed that the error in the thickness of the solidified layer lies clearly below 2·5 per cent

The process of solidification in pipe flow shows several pecularities. If solidification comes to a stop this critical point may represent a stable or an unstable state, depending on the magnitude of the parameters Bi and $Bi/Phq^* = k'(T_s - T_u)/\dot{q}$. A stable point is characterized by the fact that, following a small disturbance, for instance by a change of heat flux at the phase interface, the solid layer returns to its original thickness as soon as the disturbance disappears. An unstable point is characterized by the fact that, after the disturbance disappears, the solid layer will continue to grow or to melt according to the sign of the disturbance. Whether the point that is reached is stable or unstable depends on preceding occurances in the solidification process. The solid layer thus posesses something like a memory. These phenomena occur also in solidification of spherical shapes.

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Résumé—La solidification de fluides s'écoulant le long d'une paroi plane ou à travers un tuyau est calculée en supposant un transport de chaleur fini vers l'ambiance et un flux de chaleur imposé ou connu vers l'interface solide–liquide.

Une très bonne approximation est obtenue si l'on suppose que la distribution de température dans la phase solide est parabolique, ce qui satisfait toutes les conditions aux limites et n'est en accord avec l'équation de l'énergie qu'aux interfaces solide–liquide. Une comparaison avec des mesures sur un modèle électrique analogique et avec d'autres résultats numériques montre que l'erreur est inférieure à 2,5 %.

Dans l'écoulement en conduite, la croissance de la couche solide, pour certaines valeurs du transport de chaleur ambiant, et du flux de chaleur à l'interface fluide–solide, peut s'arrêter pour deux points critiques, dont l'une désigne des conditions stables et l'autre un étét instable. Selon l'histoire de la phase solide, on atteint un point stable ou instable.

Zusammenfassung—Die Verfestigung von Flüssigkeiten, die entlang einer ebenen Wand oder in einem Rohr strömen, wird unter der Annahme berechnet, dass der Wärmeübergang an die Umgebung endlich ist und ein aufgezwungener oder bekannter Wärmefluss an der Kontaktfläche von Flüssigkeit und Feststoff vorliegt.

Eine sehr gute Näherung erhält man, wenn in der festen Phase eine parabolische Temperaturverteilung angenommen wird, die allen Randbedingungen genügt und mit der Energiegleichung nur an der fest-flüssig-Kontaktfläche übereinstimmt. Ein Vergleich mit Messungen an einem elektrischen Analogiemodell und mit anderen numerischen Ergebnissen zeigt, dass die Abweichung unter 2,5 Prozent liegt.

Bei der Roherströmung kann das Anwachsen der festen Schicht für bestimmte Werte des Wärmeübergangs an die Umgebung und des Wärmestroms an der flüssig-festen Kontaktfläche an zwei kritischen Punkten zum Stillstand kommen, wobei der eine stabile Verhältnisse, der andere einen instabilen Zustand bezeichnet. Es hängt von der Vorgeschichte der festen Phase ab, ob ein stabiler oder instabiler Punkt erhalten wird.

Аннотация—Расчет процесса затвердевания жидкостей, движущихся вдоль плоской стенки или текущих в трубе, предлагается проводить в допущении наличия конечного переноса тепла в окружающую среду и заданного теплового потока к поверхности раздела твердое тело-жидкость. Хорошее приближение получается для параболического распределения температур в твердой фазе, что удовлетворяет всем граничным условиям и соответствует уравнению энергии только на поверхностях раздела твердое тело-жидкость. Сравнение этого расчёта с данными, полученными с помощью аналоговой электрической модели, и с другими численными результатами обнаруживает погрешность меньшую 2,5 %. Для течения в трубе нарастание твердого слоя при определенных интенсивностях теплообмена с окружающей средой и плотности потока тепла на поверхности раздела жидкость-твердое тело может приводить к остановке в двух случаях, один из которых означает стационарный режим, а другой-нестационарный, что зависит от всего предыдущего процесса.